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Compatibility and Marginality

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We present a way of introducing joint distribution function and its marginal distribution functions for non-compatible observables. Each such marginal distribution function has the property of commutativity. Models based on this approach can be used to better explain some classical phenomena in stochastic processes.

KEY WORDS: Orthomodular lattice; state; observable.

1. INTRODUCTION

Let $(\Omega, \Im, P)$ be a probability space and let ξ_1, ξ_2, ξ_3 be random variables. Then

$$F_{\xi_1,\xi_2,\xi_3} = P\left(\left\{\omega \in \Omega; \bigcap_{i=1}^{3} \xi_i^{-1}(-\infty, r_i)\right\}\right)$$

is the distribution function and the marginal distribution function is defined by the following way

$$F_{\xi_1,\xi_2}(r_1, r_2) = \lim_{x_3 \to \infty} F_{\xi_1,\xi_2,\xi_3}(r_1, r_2, r_3).$$

From the definition of a distribution function it follows, that all random variables are simultaneously measurable. It means, that they can be observable at the same time.

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Let $(\Omega_i, \Im_i, P_i)$ be probability spaces and $i = 1, 2, \ldots, n$ be the time coordinate. Let ξ_i be random variable on the probability space $(\Omega_i, \Im_i, P_i)$. How to define the joint distribution function, now?

In this paper, we will study such random events, which are not simultaneously measurable. One of the approache to this problem is studing an algebraic strucute an orthomodular lattice (an OML) (Dvurečenskij and Pulmannová, 2000; Varadarajan, 1968)[4]

Definition 1.1. Let L be a nonempty set endowed with a partial ordering $\leq$. Let there exists the greatest element 1 and the smallest element 0. We consider operations supremum ($\vee$), infimum $\wedge$ (the lattice operations) and an map $\perp$: $L \to L$ defined as follows.

(i) For any $\{a_n\}_{n\in\mathcal{A}} \in L$, where $\mathcal{A} \subset \mathcal{N}$ is finite,

$$\bigvee_{n\in\mathcal{A}} a_n, \ \bigwedge_{n\in\mathcal{A}} a_n \in L.$$

(ii) For any $a \in L (a^{\perp})^{\perp} = a$.
(iii) If $a \in L$, then $a \vee a^{\perp} = 1$.
(iv) If $a, b \in L$ such that $a \leq b$, then $b^{\perp} \leq a^{\perp}$.
(v) If $a, b \in L$ such that $a \leq b$ then $b = a \vee (a^{\perp} \wedge b)$ (orthomodular law).

Then $(L, 0, 1, \vee, \wedge, \perp)$ is said to be the orthomodular lattice (briefly the OML).

Let L be an OML. Then elements $a, b \in L$ will be called:

- *orthogonal* $(a \perp b)$ iff $a \leq b^{\perp}$;
- *compatible* $(a \leftrightarrow b)$ iff there exist mutually orthogonal elements $a_1, b_1, c \in L$ such that

$$a = a_1 \vee c \quad \text{and} \quad b = b_1 \vee c.$$

If $a_i \in L$ for any $i = 1, 2, \ldots, n$ and $b \in L$ is such, that $b \leftrightarrow a_i$ for all i, then $b \leftrightarrow \bigvee_{i=1}^{n} a_i$ and

$$b \wedge \left(\bigvee_{i=1}^{n} a_i \right) = \bigvee_{i=1}^{n} (a_i \wedge b)$$

(Dvurečenskij and Pulmannová, 2000; Varadarajan, 1968).

Let $a, b \in L$. It is easy to show, that $a \leftrightarrow b$ if and only if $a = (a \vee b) \wedge (a \vee b^{\perp})$ (distributive law). Moreover, L is a Boolean algebra if and only if L is distributive. The well known example of an OML is the lattice of orthogonal projectors in a Hilbert space.

[4] We also mention so called contextual probabilistic approach (Khrennikov, 2003a,b).

Let $(\Omega, \Im, P)$ be a probability space. Then a statement A is represented as a measurable subset of $\Omega (A \in \Im)$. For example, if we say A or B it means $A \cup B$ and non A it means A^c (the set complement in Ω).

If a basic structure is an OML, then a and b it means infinum $(a \wedge b)$, a or b it means supremum $(a \vee b)$ and non a means $a^\perp$.

If $(\Omega, \Im, P)$ is a probability space, then for any $A, B \in \Im$

$$A = (A \cap B) \cup (A \cap B^c).$$

If L is an OML, then for any $a, b \in L$

$$a \geq (a \wedge b) \vee (a \wedge b^\perp).$$

Example 1. Let L be the Hilbert space R^2. Then $1 := R^2$ and $0 := [0, 0]$. If $a \in L - \{1, 0\}$, then a is a linear subspace of R^2, it means that a is a line, which contains the point $[0, 0]$. We can write, that $a : y = k_a x$. Let $a, b \in L$, $a \neq b$. If $a : y = k_a x$, $b : y = k_b x$, then $a^\perp : y = -\frac{1}{k_a}$, $a \wedge b = [0, 0]$ and $a \vee b = R^2$.

On an OML we can define similar notions as on a measurable space $(\Omega, \Im)$.

Definition 1.2. A map m : $L \to [0, 1]$ such that

(i) $m(0) = 0$ and $m(1) = 1$.
(ii) If $a \perp b$ then $m(a \vee b) = m(a) + m(b)$
is called a state on L.

Let $\mathcal{B}(\mathcal{R})$ be a σ-algebra of Borel sets. A homomorphism $x : \mathcal{B}(\mathcal{R}) \to L$ is called an observable on L. If x is an observable, then $R(x) := \{x(E); E \in \mathcal{B}(\mathcal{R})\}$ is called a range of the observable x. It is clear that $R(x)$ is a Boolean algebra [Var]. A spectrum of an observable x is defined by the following way: $\sigma(x) = \cap\{E \in \mathcal{B}(\mathcal{R}); x(E) = 1\}$. If g is a real function, then $g \circ x$ is such observable on L that:

(1) $R(g \circ x) \subset R(x)$;
(2) $\sigma(g \circ x) = \{g(t); t \in \sigma(x)\}$;
(3) for any $E \in \mathcal{B}(\mathcal{R})$

$$g \circ x(E) = x(\{t \in \sigma(x); g(t) \in E\}).$$

We say that x and y are compatible $(x \leftrightarrow y)$ if there exists a Boolean subalgebra $\mathcal{B} \subset L$ such that $R(x) \cup R(y) \subset \mathcal{B}$. In other words $x \leftrightarrow y$ if for any $E, F \in \mathcal{B}(\mathcal{R})$, $x(E) \leftrightarrow y(F)$.

We call an observable x a finite if $\sigma(x)$ is a finite set. It means, that $\sigma(x) = \{t_i\}_{i=1}^n$, $n \in N$. Let us denote $\mathcal{O}$ the set of all finite observables on L.

A state is an analogical notion to the probability measure, an observable is analogical to a random variable.

2. S-MAP

Let L be an OML. In the papers (Nánásiová, 2003; Nánásiová and Khrennikov, 2002) is defined s-map in the following way:

Definition 2.1. Let L be an OML. The map $p : L^2 \to [0, 1]$ will be called s-map if the following conditions hold:

(s1) $p(1, 1) = 1$;
(s2) if there exists $a \perp b$, then $p(a, b) = 0$;
(s3) if $a \perp b$, then for any $c \in L$,

$$p(a \vee b, c) = p(a, c) + p(b, c)$$

$$p(c, a \vee b) = p(c, a) + p(c, b)$$

The s-map allows us e.g. to define a conditional probability for non compatible random events, a joint distribution, a conditional expectation and covariance for non compatible observales. Such random events cannot be described by the classical probabilty theory (Kolmogoroff, 1933). This problems are studed, for example in Nánásiová (1998, 2003, 2004), Nánásiová and Khrennikov (2002).

In this section we will introduce n-dimensional an s-map (briefly an s_n-map) and we will show its basic properties.

Definition 2.2. Let L be an OML. The map $p : L^n \to [0, 1]$ will be called an s_n-map if the following conditions hold:

(s1) $p(1, \ldots, 1) = 1$;
(s2) if there exist i, such that $a_i \perp a_{i+1}$, then $p(a_1, \ldots, a_n) = 0$;
(s3) if $a_i \perp b_i$ then

$$p(a_1, \ldots, a_i \vee b_i, \ldots, a_n) = p(a_1, \ldots, a_i, \ldots, a_n) + p(a_1, \ldots, b_i, \ldots, a_n),$$

for $i = 1, \ldots, n$.

Proposition 2.1. *Let L be an OML and let p be an s_n-map. Then*

(1) if $a_i \perp a_j$, then $p(a_1, \ldots, a_n) = 0$;
(2) for any $a \in L$, a map $\nu : L \to [0, 1]$, such that $\nu(a) := p(a, \ldots, a)$ is a state on L;

(3) for any $(a_1, \ldots, a_n) \in L^n$ $p(a_1, \ldots, a_n) \le \nu(a_i)$ *for each* $i = 1, \ldots, n$;
(4) if $a_i \leftrightarrow a_j$, *then*

$$p(a_1, \ldots, a_n) = p(a_1, \ldots, a_{i-1}, a_i \wedge a_j, \ldots, a_j \wedge a_i, a_{j+1}, \ldots, a_n)$$

Proof:

(1) It is enought to prove, that $p(a_1, \ldots, a_n) = 0$ if $a_1 \perp a_n$. Let $(a_1, \ldots, a_n) \in L^n$ and let $a_1 \perp a_n$. Then

$$\begin{aligned} 0 \le p(a_1, \ldots, a_n) &\le p(a_1, \ldots, a_{n-1}, a_n) + p(a_1, \ldots, a_{n-1}^{\perp}, a_n) \\ &= p(a_1, \ldots, a_{n-2}, 1, a_n) \\ &= p(a_1, \ldots, a_{n-2}, a_n, a_n) + p(a_1, \ldots, a_{n-2}, a_n^{\perp}, a_n) \\ &= p(a_1, \ldots, a_{n-2}, a_n, a_n) \le \cdots \le p(a_1, a_n, \ldots, a_n) \\ &= 0. \end{aligned}$$

From this follows, that $p(a_1, \ldots, a_n) = 0$.

(2) It is clear, that $\nu(0) = 0$, and $\nu(1) = 1$. Let $a, b \in L$, such that $a \perp b$. Then

$$\begin{aligned} \nu(a \vee b) &= p(a \vee b, \ldots, a \vee b) \\ &= p(a, a \vee b, \ldots, a \vee b) + p(b, a \vee b, \ldots, a \vee b) \\ &= p(a, a, a \vee b, \ldots, a \vee b) + p(a, b, a \vee b, \ldots, a \vee b) \\ &\quad + p(b, a, a \vee b, \ldots, a \vee b) + p(b, b, a \vee b, \ldots, a \vee b) \\ &= p(a, a, a \vee b, \ldots, a \vee b) + p(b, b, a \vee b, \ldots, a \vee b) \\ &= \ldots = p(a, \ldots, a) + p(b, \ldots, b) \\ &= \nu(a) + \nu(b). \end{aligned}$$

From it follows, that ν is a state on L.

(3) Let $(a_1, \ldots, a_n \in L^n)$. Then for any $i = 1, \ldots, n$ we have

$$p(a_1, \ldots, a_i, \ldots, a_n) \le p(a_1, \ldots, a_i, \ldots, a_n) + p(a_1^{\perp}, \ldots, a_i, \ldots, a_n).$$

and so

$$\begin{aligned} p(a_1, a_2, \ldots, a_i, \ldots, a_n) &\le p(1, a_2, \ldots, a_i, \ldots, a_n) \\ &= p(a_i, a_2, \ldots, a_i, \ldots, a_n). \end{aligned}$$

From it follows, that

$$\begin{aligned} p(a_1, a_2, \ldots, a_i, \ldots, a_n) &\le p(a_1, 1, \ldots, a_i, \ldots, a_n) \\ &= p(a_i, a_i, a_3, \ldots, a_i, \ldots, a_n). \end{aligned}$$

Hence

$$p(a_1, \ldots, a_n) \leq p(a_1, \ldots, a_i) = \nu(a_i).$$

(4) Let $a, b \in L$, such that $a \leftrightarrow b$. Then $a = (a \wedge b) \vee (a \wedge b^{\perp})$ and $b = (b \wedge a) \vee (b \wedge a^{\perp})$. Let $(a_1, \ldots, a_n) \in L^n$ and let $a_1 \perp a_2$. Then

$$p(a_1, a_2, \ldots, a_n) = p((a_1 \wedge a_2 \perp) \vee (a_1 \wedge a_2 \perp), a_2 \ldots, a_n).$$

From the property(s3) and for the property (1) we get

$$p(a_1, a_2, \ldots, a_n) = p(a_1 \wedge a_2, a_2, \ldots, a_n).$$

And hence

$$p(a_1, a_2, a_3, \ldots, a_n) = p(a_1 \wedge a_2, a_2 \wedge a_1, a_3 \ldots, a_n).$$

□

Let $\bar{a} = (a_1, \ldots, a_n) \in L^n$. Let us denote $\pi(\bar{a})$ a permutation of $(a_1, \ldots, a_n)$.

Proposition 2.2. *Let L be an OML. Let p be an s_n-map and let $(a_1, \ldots, a_n) \in L^n$.*

(1) If there exists $i \in \{1, \ldots, n\}$, such that $a_i = 1$, then

$$p(a_1, \ldots, a_n) = p(a_1, \ldots, a_{i-1} \cdot a_j, a_{i+1}, \ldots, a_n)$$

for each $j = 1, \ldots, n$.

(2) If there exist $i \neq j$ such that $a_i = a_j$, then

$$p(a_1, \ldots, a_n) = p(\pi(a_1, \ldots, a_n)).$$

(3) If there exist i, j such that $a_i \leftrightarrow a_j$, than

$$p(a_1, \ldots, a_n) = p(\pi(a_1, \ldots, a_n)).$$

Proof:

(1) Let $a_i = 1$ and let $i \neq j$. Then $a_i = a_j \vee a_j^{\perp}$ and from the Proposition 2.1(1) follows that $p(a_i, \ldots, a_{i-1}, a_j^{\perp}, a_{j+1}, \ldots a_n) = 0$. From the property (s3) we get

$$p(a_1, \ldots, 1, a_{i+1}, \ldots a_n) = p(a_1, \ldots, a_j, a_{i+1}, \ldots a_n) + p(a_1, \ldots, a_j^{\perp}, a_{i+1}, \ldots a_n).$$

And so

$$p(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) = p(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_n).$$

(2) If $n = 2$ and $a_1 = a_2$ then it is clear that $p(a_1, a_2) = p(a_2, a_1)$. Let $n \geq 3$ and let $1 \neq i$ and $i \neq n$. Let $a_1 = a_n = a$. It is enought to prove, that

$$p(a, a_2, \ldots, a_i, \ldots, a_{n-1}, a) = p(a_i, a_2, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_{n-1}, a).$$

From the (1) we have

$$p(a, a_2, \ldots, a_i, \ldots, a_{n-1}, a) = p(1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}, a).$$

From it follows, that

$$p(1, a_2, \ldots, a_i, \ldots, a_{n-1}, a) = p(a_i, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}, a)$$

and

$$p(a_i, a_2, \ldots, a_i, \ldots, a_{n-1}, a) = p(a_i, a_2, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_{n-1}, a).$$

From the aditivity it follows, that

$$p(a_i, a_2, \ldots, 1, a_{i+1}, \ldots, a_{n-1}, a) = p(a_i, a_2, \ldots, a, a_{i+1}, \ldots, a_{n-1}, a).$$

Hence

$$p(a, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{n-1}, a) = p(a_i, a_2, \ldots, a_1, a_{i+1}, \ldots, a_{n-1}, a_n).$$

From it follows that $p(a_1, \ldots, a_n) = p(\pi(a_1, \ldots, a_n))$, if there exist i, j, such that $i \neq j$ and $a_i = a_j$.

(3) Let $a_1 \leftrightarrow a_n$. Then

$$p(a_1, \ldots, a_n) = p(a_1 \wedge a_n, a_2, \ldots, a_{n-1}, a_n \wedge a_1).$$

Because $a_1 \wedge a_n = a_n \wedge a_1$ and from the property (2) it follows, that

$$p(a_1, \ldots, a_n) = p(\pi(a_1, \ldots, a_n)).$$

□

Let $\Pi(\bar{a})$ be the set of all permutions and let

$$\bar{a}^{(i)}_{(k)} = (a_1, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_{i-1}, a_k, a_{i+1}, \ldots, a_n).$$

Corollary 2.2.1. *Let L be an OML. Let p be an s_n-map and let $\bar{a} \in L^n$.*

(1) If there exists $i \in \{1, \ldots, n\}$, such that $a_i = 1$, then

$$p(\bar{a}) = p(\bar{b})$$

for each $\bar{b} \in \bigcup_k \Pi(\bar{a}^{(i)}_{(k)})$.

(2) If there exist $i \neq j$ such that $a_i = a_j$, then

$$p(\bar{a}) = p(\bar{b})$$

for each $\bar{b} \in \bigcup_k \Pi(\bar{a}^{(i)}_{(k)})$.

(3) If there exist ij such that $a_i \leftrightarrow a_j$, then

$$p(\bar{a}) = p(\bar{b})$$

for each $\bar{b} \in \bigcup_k \Pi(\bar{a}^{(i)}_{(k)})$.

Example 1 Let $n = 3$ and $a, b \in L$. If $\bar{a} = (a, a, b)$, then $\Pi(\bar{a}) = \{(a, a, b), (b, a, a), (a, b, a)\}$ and $\bar{a}^1_3 = (b, a, b), \bar{a}^2_3 = (a, b, b), \bar{a}^{(1)}_{(2)} = (a, a, b)$. Hence

$$p(a, a, b) = p(a, b, a) = p(b, a, a) = p(b, b, a) = p(a, b, b) = p(b, a, b).$$

Let $n = 4$ and $a, b, c \in L$. If $\bar{a} = (a, b, c, c)$, then $\bar{a}^{(4)}_{(2)} = (a, b, c, b)$ and

$$p(a, a, b, c) = p(a, b, c, a) = p(b, b, c, a) = \cdots = p(c, a, b, c).$$

3. THE JOINT DISTRIBUTION FUNCTION AND MARGINAL DISTRIBUTION FUNCTIONS

Definition 1 Let L be an OML and let p be an s_n-map. If $x_1, \ldots, x_2$ are observables on L, then the map

$$p_{x_1,\ldots,x_n}\colon \mathcal{B}(R)^n \to [0, 1],$$

such that

$$p_{x_1,\ldots,x_n}(E_1, \ldots, E_n) = p(x_1(E_1), \ldots, x_n(E_n))$$

is called the joint distribution of the observables $x_1, \ldots, x_n$.

Definition 1 Let L be an OML and let p be an s_n-map. If $x_1, \ldots, x_2$ be observables on L, then the map

$$F_{x_1,\ldots,x_n}\colon R^n \to [0, 1],$$

such that

$$F_{x_1,\ldots,x_n}(r_1, \ldots, r_n) = p(x_1(-\infty, r_1), \ldots, x_n(-\infty, r_n))$$

is called the joint distribution function of the observables $x_1, \ldots, x_n$.

Definition 1 Let L be an OML and let p be an s_n-map. If $x_1, \ldots, x_2$ be observables on L, then a marginal distribution function is

$$\lim_{x_i \to \infty} F_{x_1,\ldots,x_i,\ldots,x_n}(r_1, \ldots, r_i, \ldots, r_n).$$

Definition 1 Let L be an OML and let p be an s_n-map. Let $x_1, \ldots, x_2$ be observables on L and $F_{x_1,\ldots,x_n}$ be the joint distribution function of the observables $x_1, \ldots, x_n$. Then we say, that $F_{x_1,\ldots,x_n}$ has the property of commutativity if for each $(r_1, \ldots, r_n) \in R^n$

$$F_{x_1,\ldots,x_n}(r_1, \ldots, r_n) = F_{\pi(x_1,\ldots,x_n)}(\pi(r_1, \ldots, r_n)).$$

It is clear that $F_{x_1,\ldots,x_n}$ has the property of commutativity if and only if

$$p(x_1(E_1), \ldots, x_n(E_n)) = p(\pi(x_1(E_1), \ldots, x_n(E_n))),$$

for each $E_i \in \mathcal{B}(R)$, $i = 1, \ldots, n$.

Proposition 3.1. *Let L be an OML and let p be an s_n-map. Let $x_1, \ldots, x_2 \in \mathcal{O}$ and let $F_{x_1,\ldots,x_n}(r_1, \ldots, r_n)$ be the joint distribution function of the observables $x_1, \ldots, x_n$.*

(1) For each $(r_1, \ldots, r_n) \in R^n 0 \leq F_{x_1,\ldots,x_n}(r_1, \ldots, r_n) \leq 1$;
(2) If $r_i \leq s_i$, then $F_{x_1,\ldots,x_n}(r_1, \ldots, r_i, \ldots, r_n) \leq F_{x_1,\ldots,x_n}(r_1, \ldots, s_i, \ldots, r_n)$.
(3) For each $i = 1, \ldots, n$

$$\lim_{x_i \to \infty} F_{x_1,\ldots,x_n}(r_1, \ldots, r_n) = F_{x_1,\ldots,x_n}(r_1, \ldots, r_{i-1}, 1, r_{i+1}, \ldots, r_n).$$

(4) For each $i = 1, \ldots, n$

$$\lim_{r_i \to -\infty} F_{x_1,\ldots,x_n}(r_1, \ldots, r_n) = 0.$$

(5) If there exist i, j, such that $i \neq j$ and $x_i \leftrightarrow x_j$, then

$$F_{x_1,\ldots,x_n}(r_1, \ldots, r_n) = F_{\pi(x_1,\ldots,x_n)}(\pi(r_1, \ldots, r_n)).$$

Proof:

(1) It follows directly from the definition of the function $F_{x_1,\ldots,x_n}$.
(2) Let $r_i \leq s_i$. Then $(-\infty, r_i) \subseteq (-\infty, s_i)$ and so $x_i((-\infty, r_i)) \leq x_i((-\infty, r_i))$ and $x_i((-\infty, s_i)) = x_i((-\infty, r_i)) \vee x_i([r_i, s_i))$. From it follows, that

$$\begin{aligned}
&F_{x_1,\ldots,x_n}(r_1, \ldots, s_i, \ldots, r_n)\\
&\quad = p(x_1((-\infty, r_1)), \ldots, x_i((-\infty, r_i)), \ldots, x_n((-\infty, r_n)))\\
&\qquad + p(x_1((-\infty, r_1)), \ldots, x_i([r_i, s_i)), \ldots, x_n((-\infty, r_n)))\\
&\quad = F_{x_1,\ldots,x_n}(r_1, \ldots, r_i, \ldots, r_n)\\
&\qquad + p(x_1((-\infty, r_1)), \ldots, x_i([r_i, s_i)), \ldots, x_n((-\infty, r_n)))
\end{aligned}$$

and so

$$F_{x_1,\ldots,x_n}(r_1,\ldots,s_i,\ldots,r_n) \geq F_{x_1,\ldots,x_n}(r_1,\ldots,r_i,\ldots,r_n).$$

(3) Because $x_i \in \mathcal{O}$, then there exist $r_{i0} \in R$, such that for any $r \geq r_{i0}\sigma(x_i) \subseteq (-\infty, r)$ and so $x_i(-\infty, r)) = 1$. Hence

$$\lim_{r_i \to \infty} F_{x_1,\ldots,x_n}(r_1,\ldots,r_n) = F_{x_1,\ldots,x_n}(r_1,\ldots,r_{i-1},1,r_{i+1},\ldots,r_n).$$

(4) Because $x_i \in \mathcal{O}$, then there exist $r_{i0} \in R$, such that for each $r \leq r_{i0}(-\infty, r) \cap \sigma(x_i) = \emptyset$ and so $x_i(-\infty, r)) = 0$. Hence

$$\lim_{r_i \to -\infty} F_{x_1,\ldots,x_n}(r_1,\ldots,r_n) = 0.$$

(5) Because $F_{x_1,\ldots,x_n}(r_1,\ldots,r_n) = p(x_1((-\infty, r_1)),\ldots,x_n(((-\infty, r_n))$, then it follows directly from the Proposition 2.2.

□

Proposition 3.2. *Let L be an OML and let p be an s_n-map. Let $x_1,\ldots,x_n \in \mathcal{O}$ and let $F_{x_1,\ldots,x_n}(r_1,\ldots,r_n)$ be the joint distribution function. Compatibility of just two observables imply the total commutativity.*

Proof: It follows directly from the definition of the joint distribution function and from the Proposition 2.2. □

Proposition 3.3. *Let L be an OML and let $x_1,\ldots,x_n \in \mathcal{O}$. Then there exist a probability space $(\Omega, \Im, P)$ and random variables $\xi_1,\ldots,\xi_n$ on it, such that*

$$F_{x_1,\ldots,x_n}(r_1,\ldots,r_n) = F_{\xi_1,\ldots,\xi_n}(r_1,\ldots,r_n)$$

and P_{ξ_i} such that

$$P_{\xi_i}((-\infty, r)) = \nu(x_i(-\infty, r)),$$

where $r \in R$ and $i = 1,\ldots,n$ is the probability distribution of the random varaible ξ_i.

Proof: Let $\Omega = \sigma(x_1) \times \cdots \times \sigma(x_n)$ and let $\Im = 2^\Omega$. Then each $\omega = (r_1,\ldots,r_2)$ $\xi_i(\omega_1,\ldots,\omega_n) = \omega_i$. Let $A \subset \Im$ and let P: $\Im \to [0, 1]$, such that

$$P(A) = \sum_{\omega \in A} p(x_1(\xi_1(\omega)),\ldots,x_n(\xi_n(\omega))).$$

$$F_{x_1,\ldots,x_n}(r_1,\ldots,r_n) = F_{\xi_1,\ldots,\xi_n}(r_1,\ldots,r_n).$$

It is clear, that $P(\emptyset) = 0$ and $P(\Omega) = P(\sigma(x_1) \times \sigma(x_n)) = 1$. Let $A, B \in \Im$, such that $A \cap B = \emptyset$. Then

$$P(A \cup B) = \sum_{\omega \in A \cup B} p(x_1(\xi_1(\omega)), \ldots, x_n(\xi_n(\omega)))$$

and so

$$P(A \cup B) = \sum_{\omega \in A} p(x_1(\xi_1(\omega)), \ldots, x_n(\xi_n(\omega))) + \sum_{\omega \in B} p(x_1(\xi_1(\omega)), \ldots, x_n(\xi_n(\omega))).$$

From it follows that

$$P(A \cup B) = P(A) + P(B).$$

From the fact, taht Ω is the finite set follows, that P is the σ-aditive measure and so $(\Omega, \Im, P)$ has the same properties as a classical probability space and $\xi_i : \Omega \to R$ is a measurable function on it. For each $r \in R$

$$P_{\xi_1}((-\infty, r)) = P(\xi_1^{-1}(-\infty, r)) = P\big((-\infty, r) \times \sigma(x_2) \times \cdots \times \sigma(x_n)\big)$$

and then

$$P_{\xi_1}(-\infty, r) = \sum_{\omega \in (-\infty, r) \times \sigma(x_2) \times \cdots \times \sigma(x_n)} p(x_1(\xi_1(\omega)), \ldots, x_n(\xi_n(\omega))).$$

From it follows, that

$$P_{\xi_1}(-\infty, r) = P(x_1((-\infty, r)), 1, \ldots, 1) = \nu(x_1(-\infty, r)).$$

From the definition of the marginal distribution function it follows, that

$$\nu(x_i(-\infty, r)) = F_{\xi_i}(r)$$

is the distribution function for the observable ξ_i and

$$p_{x_1,\ldots,x_n}(r_1, \ldots, r_n) = F_{\xi_1,\ldots,\xi_n}(r_1, \ldots, r_n).$$

is a joint distribution for that vector of random variables $(\xi_1, \ldots, \xi_n)$. □

If we consider a quantum model as an OML, a marginal distribution function defined by using an s_n-map has the property of commutativity. It follows that, in general, it need not true that that

$$F_{x_1,\ldots,x_n}(t_1, \ldots, t_n) = F_{x_1,\ldots,x_{n+1}}(t_1, \ldots, t_n, \infty),$$

where $F_{x_1,\ldots,x_n}(t_1, \ldots, t_n)$, $F_{x_1,\ldots,x_{n+1}}(t_1, \ldots, t_n, \infty)$ are joint distribution functions and $x_1, \ldots, x_{n+1}$ are observables on L. Consequently, we can find such an s_n-map and an s_{n+1}-map such that

$$p(a_1, \ldots, a_n) \neq p(a_1, \ldots, a_n, 1)$$

on L. Moreover if

$$p(a_1, \ldots, a_n) = p(a_1, \ldots, a_n, 1)$$

on L, then the s_n-map has the property of commutativity. This is not true in general, either (Nánásiová, 2003; Nánásiová and Khrennikov, 2002).

Example 1 Let $L = \{a, a^{\perp}, b, b^{\perp}, c, c^{\perp}, 0, 1\}$. Let $x, y, z \in \mathcal{O}$. Let $\sigma(x) = \sigma(y) = \sigma(z) = \{-1, 1\}$. Let $x(1) = a$, $x(1) = b$ and $z(1) = c$. Let an s_3-map be defined by the following way:

$$p(a, a, a) = 0.3, \quad p(b, b, b) = 0.4, \quad p(c, c, c) = 0.5,$$

$$p(a, b, 1) = 0.1, \quad p(a, b^{\perp}, 1) = 0.2, \quad p(a^{\perp}, b, 1) = 0.3, \quad p(a^{\perp}, b^{\perp}, 1) = 0.4,$$

$$p(a, c, 1) = 0.2, \quad p(a, c^{\perp}, 1) = 0.1, \quad p(a^{\perp}, c, 1) = 0.3, \quad p(a^{\perp}, c^{\perp}, 1) = 0.4,$$

$$p(b, c, 1) = 0.2, \quad p(b, c^{\perp}, 1) = 0.2, \quad p(b^{\perp}, c, 1) = 0.3, \quad p(b^{\perp}, c^{\perp}) = 0.3,$$

$$p(a, b, c) = 0, \quad p(a, b, c^{\perp}) = 0.1, \quad p(a, b^{\perp}, c) = 0.2, \quad p(a, b^{\perp}, c^{\perp}) = 0,$$

$$p(a^{\perp}, b, c) = 0.2, \quad p(a^{\perp}, b, c^{\perp}) = 0.1, \quad p(a^{\perp}, b^{\perp}, c) = 0.1,$$
$$p(a^{\perp}, b^{\perp}, c^{\perp}, 1) = 0.3,$$

$$p(b, a, c) = 0.1, \quad p(b, a, c^{\perp}) = 0, \quad p(b^{\perp}, a, c) = 0.1, \quad p(b^{\perp}, a, c^{\perp}) = 0.1,$$

$$p(b, a^{\perp}, c) = 0.1, \quad p(b, a^{\perp}, c^{\perp}) = 0.2, \quad p(b^{\perp}, a^{\perp}, c) = 0.2,$$
$$p(b^{\perp}, a^{\perp}, c^{\perp}, 1) = 0.2,$$

$$p(c, a, b) = 0.01, \quad p(c, a, b^{\perp}) = 0.19, \quad p(c, a^{\perp}, b) = 0.19,$$
$$p(c, a^{\perp}, c^{\perp}) = 0.11,$$
$$p(c^{\perp}, a, b) = 0.09, \quad p(c^{\perp}, a, b^{\perp}) = 0.01, \quad p(c^{\perp}, a^{\perp}, b) = 0.11,$$
$$p(c^{\perp}, a^{\perp}, b^{\perp}, 1) = 0.29,$$

$$p(a, b, c) = p(a, c, b), \quad p(b, a, c) = p(b, c, a), \quad p(c, a, b) = p(c, b, a)$$

$$p(a^{\perp}, b^{\perp}, c^{\perp}) = p(a^{\perp}, c^{\perp}, b^{\perp}), \; p(b^{\perp}, a^{\perp}, c^{\perp}) = p(b^{\perp}, c^{\perp}, a^{\perp}),$$
$$p(c^{\perp}, a^{\perp}, b^{\perp}) = p(c^{\perp}, b^{\perp}, a^{\perp}).$$

Then p is an s_3-map and

$$F_{x_1,x_2,x_3}(1, 1, 1) = p(a^{\perp}, b^{\perp}, c^{\perp}) = 0.3,$$
$$F_{x_2,x_1,x_3}(1, 1, 1) = p(b^{\perp}, a^{\perp}, c^{\perp}) = 0.2,$$

$$F_{x_3,x_2,x_1}(1,1,1) = p(c^{\perp}, b^{\perp}, a^{\perp}) = 0.29,$$

$$\lim_{r_1\to\infty} F_{x_1,x_2,x_3}(r_1,r_2,r_3) = p(1, y(r_2), z(r_3)) = p(1, z(r_3), z(r_2))$$

$$= \lim_{r_1\to\infty} F_{x_1,x_3,x_2}(r_1,r_3,r_2),$$

where $r_2, r_3 \in R$.

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